Online Appendix

A Representative Agent Economy

A.1 RA: Environment

Notation closely follows that of the main text. There exists a representative household that chooses real consumption flows $\tilde{c}_{jt}$ to maximize

$$\int e^{-\tilde{\rho}t}\frac{\tilde{c}_{jt}^{1-\gamma}}{1-\gamma}\mathrm{d}t \tag{A.1}$$

Initial nominal assets A_0 are given. The household faces a flow budget constraint

$$dA_t = [i_t A_t + (1-\tau_t)P_t y_t - P_t\tilde{c}_t]dt \tag{A.2}$$

subject to the borrowing constraint $A_t \geq 0$, where τ_t is a path of taxes set by the government. We may express the budget constraint in real de-trended terms as

$$da_t = [r_t a_t + (1-\tau_t) - c_t]dt \tag{A.3}$$

where the real rate is defined as $r_t := i_t - \pi_t - g$. Government debt dynamics follow

$$db_t = [r_t b_t - \tau_t]dt \tag{A.4}$$

We also impose the commonly maintained assumption in the fiscal theory of the price level that the government can borrow, but not lend: $b_t \geq 0$.

Household Optimality. It is easy to show that the solution to the representative household problem yields the Euler equation

$$\rho - r_t = -\gamma\frac{1}{c_t(a_t)}\frac{dc_t(a_t)}{dt} \tag{A.5}$$

together with the household's transversality condition

$$\lim_{t\to\infty} e^{-\rho t}c^{-\gamma}a_t \leq 0 \tag{A.6}$$

Monetary Policy. We allow for arbitrary monetary policy rules i_t, but assume that they lead to well-defined paths for inflation given real rates r_t (see Section 2.3).

A.2 RA: Equilibrium Definition

We now define a real equilibrium under the assumption that the price level P_t is differentiable for all $t > 0$.

Definition 2. *A real equilibrium is a collection of variables* $\{c_t, a_t, b_t, r_t\}_{t\geq 0}$ *such that:*

1. *For all* $t > 0$, c_t *satisfies the Euler equation* (A.5) *and transversality condition* (A.6).
2. *For all* $t > 0$, a_t *evolves according to the budget constraint* (A.3).
3. *For all* $t > 0$, b_t *evolves according to the government budget constraint* (A.4).
4. *For all* $t \geq 0$, *markets clear:* $a_t = b_t$.

Note that by Walras' Law, $c_t = 1$ for all $t \geq 0$ so that the goods market clears.

A.3 RA: Uniqueness With Constant Surpluses

Next, we show that a real unique equilibrium exists whenever $\tau = \tau^* > 0$, so that the government is running constant surpluses. First, note that the Euler equation (A.5) along with market clearing for output $c_t = 1$ implies that $r_t = \rho$ for all $t \geq 0$. Integrating the government budget constraint forwards then yields (A.4):

$$b_0 = \lim_{T\to\infty}\left[\int_0^T e^{-\rho t}\tau^* \mathrm{d}t + e^{-\rho T} b_T\right] \tag{A.7}$$

By transversality (A.6) and market clearing, the latter term must be non-positive. Moreover, it cannot be negative as this would violate the non-negativity constraint on household assets and/or the assumption that the government cannot be a lender. Hence, it must be zero. But this then implies that

$$b_0 = \lim_{T\to\infty}\left[\int_0^T e^{-\rho t}\tau^* \mathrm{d}t\right] = \frac{\tau^*}{\rho} \tag{A.8}$$

so b_0 is well-defined and strictly positive for any level of initial nominal assets B_0. The dynamics for real debt $\{b_t\}_{t>0}$ are then pinned down by the government budget constraint (A.4). This proves the existence of a unique real equilibrium.

Given an initial level of nominal debt B_0, uniqueness of the real equilibrium implies uniqueness of the initial price level P_0. Subsequent inflation is uniquely pinned down by $r_t = \rho$, and a monetary policy rule which sets the path for the nominal rate i_t.

The Case of Deficits. The analysis above requires that the present discounted value in (A.8) be finite and positive. Hence, running persistent deficits cannot be an admissible equilibrium under the assumption that (i) households face borrowing constraints or (ii) that aggregate government debt must be non-negative.

B Representative Agent with Bonds-In-Utility

B.1 RA-BIU: Environment

Our notation follows closely that of the main text. Time is continuous and indexed by t. The economy is populated by a representative agent that derives utility from consumption streams c_t and real asset holdings a_t according to:

$$\int e^{-\rho t}\left(\frac{c_t^{1-\gamma}}{1-\gamma}+\zeta\ln(a_t+\underline{a})\right)\mathrm{d}t \tag{B.1}$$

where $\rho > 0$ denotes the household's discount rate, and ζ, $\underline{a}$ are positive constants. Our assumption that real assets enter utility in a logarithmic fashion is inessential to the main results. However, logarithmic utility will allow us to characterize the steady-states of the economy in closed-form.

As derived in Section A.1 at all points in time in which the price level is differentiable, the household budget constraint can be written in real terms as follows:

$$da_t = [r_t a_t + (1-\tau_t)y_t - c_t]\,dt \tag{B.2}$$

where $r_t = i_t - \pi_t$ denotes the real interest rate on bonds. The government budget constraint can similarly be written in real terms as:

$$db_t = [r_t b_t - \tau_t y_t]dt \tag{B.3}$$

We also impose the commonly maintained assumption in the fiscal theory of the price level that the government can borrow, but not lend: $b_t \geq 0$.[39] To simplify the exposition, in this section only, we assume zero growth. Positive growth is straightforward to incorporate by letting ζ grow over time at the appropriate rate.

[39]As explained in the context of the RA model of Section A, this can also be rationalized through a borrowing constraint on the household side.

Household Optimality. The representative household takes the future sequence of real rates r_t and output y_t as given, and chooses consumption and real asset holdings optimally subject to its budget constraint (B.2). This implies the following Euler Equation

$$\frac{1}{c_t}\frac{dc_t}{\mathrm{d}t} = \frac{1}{\gamma}\left(r_t - \rho + \frac{\zeta c_t^{\gamma}}{a + \underline{a}}\right) \tag{B.4}$$

The household must also satisfy the following transversality condition:

$$\lim_{t\to\infty} e^{-\rho t} c_t^{-\gamma} a_t \leq 0 \tag{B.5}$$

B.2 RA-BIU: Equilibrium Definition

The definition of equilibrium for this model is exactly as in Section A.2, with the exception that the Euler equation is given by (B.4).

Price Level Determination. As in the main text, each real equilibrium defines a unique price level determined by:

$$P_0 = \frac{B_0}{b_0} \tag{B.6}$$

The path of inflation is then determined residually through the Fisher identity $\pi_t = i_t - r_t$. We assume for simplicity a monetary policy peg, $i_t = i^*$, but note that all our results on equilibrium uniqueness extend to the more general monetary rules outlined in Section 2.3.

B.3 RA-BIU: Uniqueness with Constant Surpluses

We now show that a unique real equilibrium exists under a constant, strictly positive surplus rule $\tau_t = \tau^*$, where $\tau^* > 0$.

Proposition 1. *A unique real equilibrium exists. Moreover, $r_t = r^*$ and $b_t = b^*$ for all $t \geq 0$, where r^* and b^* are strictly positive constants that are given by:*

$$r^* = \begin{cases} \frac{-(\tau^*+\zeta-\rho\underline{a})+\sqrt{(\tau^*+\zeta-\rho\underline{a})^2+4\rho\underline{a}\tau^*}}{2\underline{a}} & \text{if} \quad \underline{a} > 0 \\ \frac{\rho\tau^*}{\tau^*+\zeta} & \text{if} \quad \underline{a} = 0 \end{cases} \tag{B.7}$$

and

$$b^* = \frac{\tau^*}{r^*} \tag{B.8}$$

Proof. Our proof proceeds in several steps.

Step 1: Monotonicity of real assets. We first show that a_t is increasing if $a_t > a^*$ and decreasing if $a_t < a^*$, where $a^* > 0$ is the unique steady-state value of real debt. We then show that this implies that $a_t < a^*$ at any t violates the non-negativity constraint on debt. Finally, we show that $a_t > a^*$ at any t is inconsistent with household optimality.

First, note that equation (B.4) together with $c_t = 1$ at all t implies that the real rate is given by the following equation for all t

$$r_t = \rho - \zeta \frac{1}{a_t + \underline{a}} \tag{B.9}$$

Imposing market clearing and using the government budget constraint (B.3), we can derive an expression for the dynamics of real debt

$$\dot{a}_t = \left(\rho - \zeta \frac{1}{a_t + \underline{a}}\right) a_t - \tau^* \tag{B.10}$$

where $\dot{a}_t \equiv \frac{da_t}{dt}$. The steady-states of this differential equation are given by

$$\frac{\tau^*}{a^*} = \rho - \zeta \frac{1}{a^* + \underline{a}} \tag{B.11}$$

Note that the left-hand side of the above equation is decreasing in a^* whenever $\tau^* > 0$ (and asymptotes to zero as $a^* \to \infty$ and infinity as $a^* \to 0$), while the right-hand side is increasing in a^* (and asymptotes to $\rho > 0$ as $a^* \to \infty$). Moreover, both terms are continuous for $a^* > 0$. Hence, a unique steady-state with a strictly positive real rate exists. Denote this real rate by $r^* > 0$.

Further, $\dot{a}_t$ is strictly positive whenever $a_t \in (a^*, \infty)$ and strictly negative whenever $a_t \in (0, a^*)$. Suppose otherwise. We have that:

$$\frac{d\dot{a}_t}{da}|_{a_t = a^*} = r^* + \zeta a^* (a^* + \underline{a})^{-2} > 0 \tag{B.12}$$

Moreover, $\dot{a}_t$ is continuously differentiable on $a_t > 0$. Hence, $\dot{a}_t(a_t) < 0$ for some $a_t \in (a^*, \infty)$ would imply that there exists an $a^{**} \in (a^*, \infty)$ such that $\dot{a}_t(a^{**}) = 0$, thereby violating steady-state uniqueness.

Step 2: Ruling Out Downwards Explosions. Next, we rule out all equilibria in which $a_{t'} < a^*$ for some $t' \geq 0$. By Step 1, $a_{t'} < a^*$ implies that $a_{t'} < a_t$ for all $t' \geq t$. Moreover, there are no limit points such that $\lim_{t\to\infty} a_t = a^{**}$ for any $a^{**} > 0$. Hence, any path in which $a_{t'} < a^*$ implies that the constraint $a_t > 0$ must be violated in finite time.

Step 3: Ruling Out Upwards Explosions. We now rule out equilibria in which $a_{t'} > a^*$ for some $t' \geq 0$. We may integrate the government budget constraint (B.3) forwards to obtain

$$a_0 = \lim_{T\to\infty} \left[\int_0^T \exp\left(\int_0^s -r_u \mathrm{d}u \right) \tau^* \mathrm{d}s + \exp\left(\int_0^T -r_u \mathrm{d}u \right) a_T \right] \tag{B.13}$$

Note that $r_t > 0$ whenever $a_t > a^*$, so the first term in the limit is well-defined. Moreover, (B.3) implies that assets will be growing at rate r_t whenever $a_t \geq a^*$. Hence, the second-term is non-zero if and only if $a_{t'} \geq a^*$ for some $t' \geq 0$.

We now show that household optimality implies that this second term must necessarily be finite. Substituting for the real rate, we obtain:

$$\lim_{T\to\infty} \left[\exp(-\rho T) a_T \times \exp\left(\int_0^T \zeta \frac{1}{a_u + \underline{a}} \mathrm{d}u \right) \right] \tag{B.14}$$

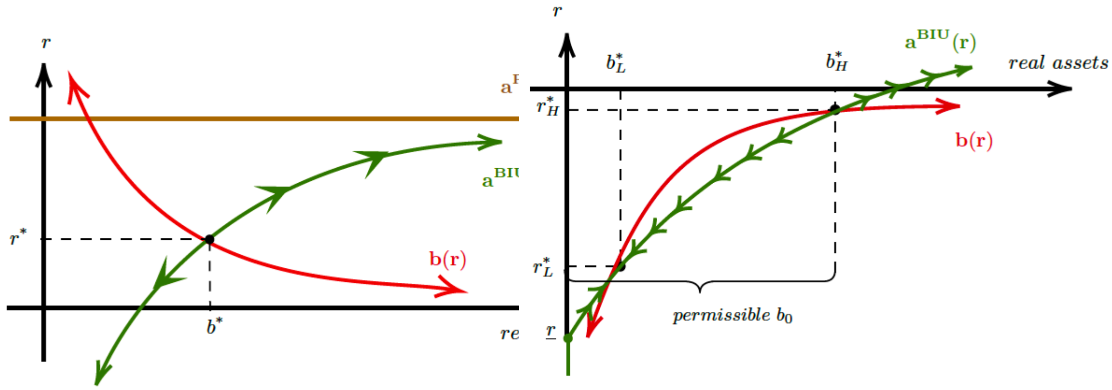
The first-term in this expression is zero by the transversality condition (B.5). The second term is bounded as assets are growing at an exponential rate. Hence, we must have $a_t = a^*$ and $r_t = r^*$ for all $t \geq 0$. Equation (B.13) then implies that the second term is zero and thus a_0 must be given by

$$a_0 = \int_0^\infty \exp\left(\int_0^s -r^* \mathrm{d}u \right) \tau^* \mathrm{d}s = \frac{\tau^*}{r^*} = a^* \tag{B.15}$$

Substituting for a^* in (B.9) yields a quadratic equation with a unique, strictly positive root given by (B.7). This completes the proof.

□

The intuition for this result closely mirrors that of the representative agent economy. The system of equations (B.3) and (B.4) are globally unstable. Paths in which $b_0 < b^*$ therefore lead to downward explosions, which violate the non-negativity condition on debt. Paths in which $b_0 > b^*$ lead to an excessive accumulation of assets,



(a) Surpluses **(b)** Deficits

Note: Dynamics for real assets in the RA-BIU economy when $\tau^* > 0$ (left panel) and $\tau^* < 0$ (right panel), as given by (B.3) and (B.4)

thereby violating household optimality. These dynamics are graphically depicted in Figure 13a. Note that r^* is strictly increasing in τ^*, with $r^* \to \rho$ and $a^* \to \infty$ as $\tau^* \to \infty$. In this sense, the steady-state asset demand in the RA-BIU economy (B.18) exhibits many similar features to the heterogeneous agent economy considered in the main text.

B.4 RA-BIU: Dynamics with Constant Deficits

Next, we consider dynamics under constant deficits $\tau^* < 0$. We show that the price level is generally no longer determinate for a given value of initial nominal debt. Intuitively, the steady-states of the government accumulation equation (B.3) form an *upward* sloping locus in $r - b$ space, as depicted graphically in Figure 13b. This can give rise to steady-state multiplicity, eliminating the explosive dynamics that are required in order to obtain uniqueness. The following proposition formally characterizes the nature of this steady-state multiplicity.

Proposition 2. *Suppose $\rho \underline{a} < \zeta$. Then:*

1. *If $\underline{a} = 0$, a unique steady-state exists if $\tau^* \in (-\zeta, 0)$, and no steady-state exists if $\tau^* \in (-\infty, -\zeta]$.*
2. *If $\underline{a} > 0$, there exists a $\underline{\tau} \in (\rho \underline{a} - \zeta, 0)$ such that two distinct steady-states exist if $\tau^* \in (\underline{\tau}, 0)$, no steady-state exists if $\tau^* \in (-\infty, \underline{\tau})$, and a unique steady-state exists if $\tau^* = \underline{\tau}$*

Proof. We may substitute for steady-state assets in (B.9) to obtain

$$r^* = \rho - \frac{\zeta}{\frac{\tau^*}{r^*} + \underline{a}} \tag{B.16}$$

We may solve the above equation to express the steady-states of the system as:

$$r^* = \begin{cases} \frac{-(\tau^*+\zeta-\rho\underline{a})\pm\sqrt{(\tau^*+\zeta-\rho\underline{a})^2+4\rho\underline{a}\tau^*}}{2\underline{a}} & \text{if} \quad \underline{a} > 0 \\ \frac{\rho\tau^*}{\tau^*+\zeta} & \text{if} \quad \underline{a} = 0 \end{cases} \tag{B.17}$$

and

$$a^* = \frac{\tau^*}{r^*} \tag{B.18}$$

where we additionally require $r^* < 0$ so that the non-negativity constraint on assets is not violated. It is straightforward to see that this condition is satisfied if and only if $\tau^* > -\zeta$ when $\underline{a} = 0$. This proves the first part of the proposition.

To prove the second part of the proposition, note a necessary and sufficient condition for $r^* < 0$ in the constant deficit economy is $\tau^* \in (\rho\underline{a} - \zeta, 0)$ and that $(\tau^* + \zeta - \rho\underline{a})^2 + 4\rho\underline{a}\tau^* > 0$. This is negative at $\tau^* = \rho\underline{a} - \zeta$, positive at $\tau^* = 0$, and strictly increasing on $(\rho\underline{a} - \zeta, 0)$. Hence, there exists a unique root of this expression within this interval given by $\underline{\tau} \in (\rho\underline{a} - \zeta, 0)$. It follows that are two distinct steady-states whenever $\underline{\tau} < \tau^* < 0$, no steady-states whenever $\tau^* < \underline{\tau}$ and a unique-steady state whenever $\tau^* = \underline{\tau}$. □

The condition $\rho\underline{a} < \zeta$ ensures that there exists a negative interest rate such that households are willing to hold strictly positive amounts of real assets (no steady-state with deficits exists trivially if this condition is not satisfied). Note also that, depending on the value of $\underline{a}$, zero, one, or two equilibria can exist. Further, at least one equilibrium exists as long as the level of deficits is not too large. We next show how steady-state multiplicity is tied to price level determinacy. In particular, a unique equilibrium exists if and only if a unique steady-state exists.

Proposition 3. *The following statements are true.*

1. *If no steady-state exists, then no real equilibria exist.*
2. *If a unique steady-state exists, then there exists a unique real equilibrium with constant real rates $r_t = r^*_H = r^*_L$ and real assets $b_t = b^*_H = b^*_L$.*

3. *If two distinct steady-states exist, then there exists a continuum of real equilibria indexed by* $b_0 \in (0, b_H^*]$.

Proof. Suppose no steady-states exist. Equation (B.3) then implies that real assets will tend to infinity or minus infinity for any given b_0. The former case is ruled out, as it violates the transversality condition by Proposition 1. The latter case is ruled out as it implies that assets will violate their non-negativity constraint in finite time. Hence, no equilibria exist.

Next, suppose that a unique steady-state exist. Define the function

$$r(a_t) = \rho - \frac{\zeta}{a_t + \underline{a}} \tag{B.19}$$

From (B.3), steady-states are given by the roots to

$$g(a) = r(a) - \frac{\tau^*}{a} \tag{B.20}$$

There exists a unique a^* such that $g(a^*) = 0$ by assumption. Moreover, $g(a) \to \rho > 0$ as $a \to \infty$, so we must have $g'(a^*) > 0$ by the intermediate value theorem. Using the government accumulation equation, the dynamics of real debt around a^* are given by

$$\begin{aligned} d\frac{\tilde{a}_t}{dt} &= [r'(a^*)a^* + r(a^*)]\tilde{a}_t \\ &= [r'(a^*) + \frac{\tau^*}{(a^*)^2}]\frac{\tilde{a}_t}{a^*} = g'(a^*)\frac{\tilde{a}_t}{a^*} > 0 \end{aligned}$$

where $\tilde{a}_t = a_t - a^*$, to first-order. Because a^* is unique by assumption, real assets explode upwards exponentially at a rate r_t when $a_0 > a^*$ (violating (B.5)) and downwards when $a_0 < a^*$ (violating the non-negativity of assets in finite time). Hence, a unique equilibrium exists.

Suppose now that two equilibria exist $a_H^* > a_L^*$. The top equilibrium is locally unstable by the argument presented above. The bottom equilibrium is locally stable, since $g(a) \to \infty$ as $a \to \infty$. Hence, $g'(a_L^*) < 0$. This implies that all equilibria with $b_0 \in (0, b_H^*)$ converge to b_L^*, while all equilibria with $b_0 > b_H^*$ feature explosive dynamics that violate (B.5). Thus, there exist a continuum of equilibria indexed by $b_0 \in (0, b_H^*]$. $\square$

One can show that the presence of two steady-states imply a non-singular basin of attraction for the economy. Hence, a continuum of real equilibria, indexed by their initial condition b_0, are possible. Note that the final condition places a lower bound on the price level for any given level of initial nominal assets, given by $P_0 = \frac{B_0}{b_H^*}$.

C Additional Derivations

C.1 Derivation of Optimal Consumption Dynamics

This section derives expressions for the consumption dynamics of unconstrained constrained households.

Unconstrained Households. We show that the expected consumption dynamics for unconstrained households are given by

$$\frac{\mathbb{E}_t\left[dc_{jt}\right]}{c_{jt}} = \frac{1}{\gamma}\left(r_t - \rho\right)\mathrm{d}t + \frac{1}{\gamma}\sum_{z'}\lambda_{z_{jt}z'}\left(\frac{c(a_{jt}, z', \Omega_t)}{c_{jt}}\right)^{-\gamma}\mathrm{d}t + \sum_{z'}\lambda_{z_{jt}z'}\left(\frac{c(a_{jt}, z', \Omega_t)}{c_{jt}}\right)\mathrm{d}t. \tag{C.1}$$

Here we use the short-hand notation $c_{jt} := c(a_{jt}, z_{jt}, \Omega_t)$ to denote the consumption of household j at time t. Recall the HJB Equation:

$$\rho V_t(a, z) = \max_c u(c) + s_t(a, z)\partial_a V(a, z) + \sum_{z' \neq z}\lambda_{z,z'}[V_t(a, z') - V_t(a, z)] + \partial_t V_t(a, z) \tag{C.2}$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and $s_t(a, z)$ is the savings function (11). The FOC is:

$$u'(c) = \partial_a V_t(a, z) \tag{C.3}$$

Differentiating the above with respect to a yields

$$u''(c_t(a, z))\partial_a c_t(a, z) = \partial^2_{aa} V_t(a, z) \tag{C.4}$$

Differentiating with respect to t yields

$$u''(c_t(a, z))\partial_t c_t(a, z) = \partial^2_{at} V_t(a, z) \tag{C.5}$$

The envelope condition for (C.2) is:

$$\rho\partial_a V_t(a,z) = \partial^2_{aa}V_t(a,z)s_t(a,z) + r_t\partial_a V_t(a,z) + \sum_{z'\neq z}[\partial_a V_t(a,z') - \partial_a V_t(a,z)] + \partial^2_{at}V_t(a,z) \tag{C.6}$$

Using (C.4) and (C.5) into the equation above yields:

$$\begin{aligned}(\rho - r_t)u'(c_t(a,z)) = &\sum_{z'\neq z}\lambda_{zz'}[u'(c_t(a,z')) - u'(c_t(a,z))] \\ &+ u''(c_t(a,z))[\partial_t c_t(a,z) + s_t(a,z)\partial_a c_t(a,z)]\end{aligned} \tag{C.7}$$

(C.7) holds at any point on the interior of the state space $a > 0$ (i.e. for all unconstrained households). Using Ito's lemma for jump processes, we can write it as:

$$(\rho - r)u'(c_t(a_j, z_j)) = \frac{d\mathbb{E}[u'(c_t(a_j, z_j))]}{dt} \tag{C.8}$$

where we suppress the dependence of a_{jt} and z_{jt} on t for notational simplicity. Furthermore, using Ito's lemma on $c_t(a_j, z_j)$ yields

$$\begin{aligned}dc_j = &\left[\partial_a c_t(a_j,z_j)s_t(a_j,z_j) + \partial_t c_t(a_j,z_j) + \sum_{z'\neq z_j}\lambda_{z_j z'}[c_t(a_j,z') - c_t(a_j,z_j)]\right]dt \\ &+ [c_t(a_j,z') - c_t(a_j,z_j)]d\tilde{N}_j\end{aligned} \tag{C.9}$$

where $\tilde{N}_j$ is the compensated Poisson process for the stochastic process of income z'. Expected consumption therefore follows:

$$\mathbb{E}[dc_j] = \left[\partial_a c_t(a_j,z_j)s_t(a_j,z_j) + \partial_t c_t(a_j,z_j) + \sum_{z'\neq z_j}\lambda_{z_j z'}[c_t(a_j,z') - c_t(a_j,z_j)]\right]dt \tag{C.10}$$

We may combine this with (C.7) to obtain

$$\begin{aligned}(\rho - r_t)u'(c_t(a_j,z_j)) = &\sum_{z'\neq z_j}\lambda_{z_j z'}[u'(c_t(a_j,z')) - u'(c_t(a,z_j))] \\ &+ u''(c_t(a_j,z_j))\frac{\mathbb{E}[dc_j]}{dt} - u''(c_t(a_j,z_j))\sum_{z'\neq z_j}\lambda_{z_j z'}[c_t(a_j,z') - c_t(a_j,z_j)]\end{aligned} \tag{C.11}$$

This yields (C.1) after dividing by $u''(c_t(a_j, z_j))$ and specializing to $u'(c) = \frac{c^{1-\gamma}}{1-\gamma}$

Constrained Households. We show that the expected consumption dynamics for borrowing constrained households satisfy:

$$\frac{\mathbb{E}_t\left[dc_{jt}\right]}{c_{jt}} = \sum_{z'} \lambda_{z_{jt},z'} \left(\frac{c(a_{jt}, z', \Omega_t)}{c_{jt}}\right) \mathrm{d}t. \tag{C.12}$$

The consumption dynamics for constrained households are given by

$$dc_t(0, z_j) = \sum_{z' \neq z_j} \lambda_{z_j z'}[c_t(0, z') - c_t(0, z_j)]dt + [c_t(0, z') - c_t(0, z_j)]d\tilde{N}_j \tag{C.13}$$

since households consume their income whenever constrained (until receiving a more favourable income draw). Taking expectations and dividing by $c_t(0, z_j)$ then yields (C.12) directly.

C.2 Existence of $\underline{r}$

This subsection shows that there exists a finite $\underline{r}$ such that no household saves in a stationary equilibrium if $r \leq \underline{r}$. Suppose no such $\underline{r}$ exists. Note that this implies that there must exist a non-zero mass of households that are unconstrained in any stationary equilibrium, for all $r < \rho$.

Proposition 2 in Achdou et al. (2022) shows that there exists a finite upper bound on the state space for assets in a stationary equilibrium. Moreover, $\underline{z} > 0$. Hence, marginal utility and consumption are bounded from above and are strictly greater than zero for all a_{jt} and z_{jt}. Equation (C.1) then implies that there must exist an $\underline{r}$ such that

$$\frac{\mathbb{E}[dc_t(a_{jt}, z_{jt})]}{dt} < 0$$

for all households j that are unconstrained. But this would then imply that aggregate consumption must be decreasing, which would violate market clearing. Hence, there cannot exist a non-zero mass of households that are unconstrained in a stationary equilibrium with $r < \underline{r}$. But this implies the existence of such an $\underline{r}$, a contradiction.

C.3 Uniqueness with Constant Surpluses

In this section, we show that explosive paths for real assets are ruled out by the household transversality condition. Our proof strategy entails decomposing the expectation in (24) and aggregating across households to show that the rate of growth of aggregate assets is bounded below by the discount rate ρ.

Consider a strictly positive sequence of real rates $(r_t)_{t\geq 0}$. Recall that the transversality condition in the stochastic economy is:

$$\lim_{t\to\infty} \left[\mathbb{E}_t \exp(-\rho t) u'(c_t(a_{jt}, z_{jt}))a_{jt}\right] = 0 \tag{C.14}$$

The household Euler equation gives us a differential equation for the evolution of *expected* marginal utility.

$$\frac{\mathbb{E}_0[du'(c_t(a_{jt}, z_{jt}))]}{u'(c_0(a_{j0}, z_{j0}))} = (\rho - r_t)\mathrm{d}t \tag{C.15}$$

We may solve this ordinary differential equation to obtain

$$\mathbb{E}_0[u'(c_t(a_{jt}, z_{jt}))] = u'(c_0(a_{j0}, z_{j0})) \exp\left(\rho t - \int_0^t r_s \mathrm{d}s\right) \tag{C.16}$$

We next decompose the expectation term in the household transversality condition:

$$\begin{aligned}\lim_{t\to\infty} \left[\mathbb{E}_0 \exp(-\rho t) u'(c_t(a_{jt}, z_{jt}))a_{jt}\right] = \lim_{t\to\infty} &[\exp(-\rho t)\mathbb{E}_0 \left[u'(c_t(a_{jt}, z_{jt}))\right] \mathbb{E}_0 \left[a_{jt}\right] \\ &+ \exp(-\rho t) Cov_0(u'(c_t(a_{jt}, z_{jt})), a_{jt})]\end{aligned} \tag{C.17}$$

where the covariance is conditional on the households' time-zero information set. We may substitute for the first term using (C.16) to obtain:

$$\lim_{t\to\infty} \left[\exp\left(-\int_0^t r_s \mathrm{d}s\right) u'(c_0(a_{j0}, z_{j0}))\mathbb{E}_0 \left[a_{jt}\right] + \exp(-\rho t)\mathrm{Cov}_0(u'(c_t(a_{jt}, z_{jt})), a_{jt})\right] = 0 \tag{C.18}$$

We may also bound the covariance term via the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}\exp(-\rho t)\mathrm{Cov}_0(u'(c_t(a_{jt}, z_{jt})), a_{jt})| &\leq \exp(-\rho t)\sqrt{\mathrm{Var}_0(u'(c_t(a_{jt}, z_{jt}))}\sqrt{\mathrm{Var}_0(a_{jt})} \\ &\leq \exp(-\rho t)\frac{y_{min}^{-\gamma}}{2}\sqrt{\mathbb{E}_0[(a_{jt})^2]}\end{aligned}$$

where last the inequality has made use of the fact that $u'(c_{jt}) \leq y_{min}^{-\gamma}$ and the Popoviciu bound on variances (Bhatia and Davis, 2000). Finally, we provide a bound on the variance of individual asset holdings. If asset holdings are uniformly bounded, the bound is trivially zero. So we only need to concern ourselves with cases in which individual assets may diverge to infinity. In these cases, we can use standard results on the asymptotic behaviour of the consumption function to provide an upper bound on assets (Benhabib et al., 2015; Achdou et al., 2022). In particular, we have:

$$\lim_{a_{jt} \to \infty} \frac{\phi_t a_{jt}}{c_{jt}} = 1 \tag{C.19}$$

where $\phi_t > 0$. We may then use the household budget constraint to show that assets grow at a rate $r_t - \phi_t$ asymptotically, which yields the bound

$$a_{jt} \leq \Xi \exp\left(\int_0^t (r_s - \phi_s) \mathrm{d}s \right), \quad \text{a.s.} \tag{C.20}$$

for some finite $\Xi > 0$. Using the Popoviciu inequality once again, we obtain

$$|\exp(-\rho t) \mathrm{Cov}_0(u'(c_{jt}), a_{jt})| \leq \exp(-\rho t) \frac{y_{min}^{-\gamma}}{4} \Xi \exp\left(\int_0^t (r_s - \phi_s) \mathrm{d}s \right) \tag{C.21}$$

Under the assumption that there exists some $t' > 0$ such that $r_t \leq \rho$ for $t \geq t'$, the right-hand side vanishes as we take $t \to \infty$. Section G provides sufficient condition for $r_t < \rho$ for all $t \geq 0$.

We now show that (C.18) precludes explosive paths for real aggregate debt. In particular, we show that

$$\lim_{t \to \infty} \left[\exp\left(\int_0^t -r_s \mathrm{d}s \right) a_t \right] = 0 \tag{C.22}$$

where a_t is the amount of aggregate asset holdings in the economy at time t. To this end, we integrate (C.18) over households to obtain:

$$\lim_{t\to\infty}\left[\int_{a,z}\mathbb{E}_0\exp(-\rho t)u'(c_0(a,z))a\mathrm{d}G_t(a,y)\right]$$
$$\leq \lim_{t\to\infty} m\left[\exp\left(-\int_0^t r_s\mathrm{d}s\right)\underbrace{\int_{a,y}\mathbb{E}_{a_0=a}\left[a_t\right]\mathrm{d}G_t(a,z)}_{\bar{a}_t}\right]=0$$

where m is an upper bound on marginal utility at $t=0$: $u'(c_0^i(a^i))\leq m \quad \forall a,y\in \text{supp } G_0(a,y),$ a.e. and where $G_t(\cdot,\cdot)$ is the distribution over assets and income at time t. Note that term in the integral in the second inequality is equal to aggregate asset holdings by the exact law of large numbers (Duffie and Sun, 2012). This shows that no equilibria exist in which government debt explodes upwards. Downward explosion paths are ruled out by the non-negativity constraint on aggregate real debt.

C.4 Finite Difference Approximation

We begin by deriving the Kolmogorov Forward Equation (KFE) for wealth shares. Note that the dynamics for wealth shares $\omega_{jt}=\frac{a_{jt}}{a_t}$ is given by

$$\frac{\mathrm{d}\omega_{jt}}{\omega_{jt}\mathrm{d}t}=\frac{\mathrm{d}a_{jt}}{a_{jt}\mathrm{d}t}-\frac{\mathrm{d}b}{b_t\mathrm{d}t} \tag{C.23}$$

Using Equations (6) and (18) yields

$$\frac{\mathrm{d}\omega_{jt}}{\mathrm{d}t}=\omega_{jt}\left(\frac{r_ta_{jt}+z_{jt}-\tau_t(z_{jt})-c_{jt}}{a_{jt}}-\frac{r_tb_t-s_t}{b_t}\right) \tag{C.24}$$

$$\frac{\mathrm{d}\omega_{jt}}{\mathrm{d}t}=\frac{z_{jt}-\tau_t(z_{jt})-c_{jt}+\omega_{jt}s_t}{b_t} \tag{C.25}$$

This implies that the KFE for wealth shares is given by:

$$\partial_t f(\omega,z)=\mathcal{A}_\omega^*[f,b](\omega,z)+\mathcal{A}_z^*[f](z) \tag{C.26}$$

where

$$\mathcal{A}_\omega^*[f,b](\omega,z) = \partial_\omega \left[f(\omega,z) \frac{z - \tau_t(z) - c_t(\omega,z;f,b) + \omega s_t}{b} \right] \tag{C.27}$$

and

$$\mathcal{A}_z^*[f](z) = -f(\omega,z) \sum_{z' \neq z} \lambda_{zz'} + \sum_{z' \neq z} \lambda_{z'z} f(\omega,z') \tag{C.28}$$

where we have made the dependence of the consumption function on aggregate state variables explicit. Note further that these operators are adjoint to underlying operators $\mathcal{A}_\omega$ and $\mathcal{A}_z$.

We may discretize the distribution $f(\omega,z)$ into $N = N_\omega \times N_z$ discrete points, where N_ω is a discrete grid for ω of width Δ_ω. We denote the discretized distribution as $\mathfrak{f}$ and write the dynamics of the joint system as

$$\frac{\mathrm{d}\mathfrak{f}}{\mathrm{d}t} = \mathbf{A}_\omega \left[\mathfrak{f}_t, b_t\right]^T \mathfrak{f}_t + \mathbf{A}_z^T \mathfrak{f}_t \tag{C.29}$$

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \mathbf{r}\left[\mathfrak{f}_t, b_t\right] b_t - s^* \tag{C.30}$$

The interest rate functional $\mathbf{r}\left[\mathfrak{f}_t, b_t\right]$ corresponds to the interest rate functional in Equation (25) where we have substituted for the discretized endowment share distribution. The matrix $\mathbf{A}_\omega\left[\mathfrak{f}_t, b_t\right]$ is a finite difference approximation to $\mathcal{A}[f,b]$ using the appropriate upwind scheme (Achdou et al., 2022). Hence, it is a tridiagonal matrix which consists of the following terms:

$$\left\{ 0, -\frac{z - \tau_t(z) - c_t(\omega,z;\mathfrak{f},b) + \omega s_t}{b\Delta_\omega}, \frac{z - \tau_t(z) - c_t(\omega,z;\mathfrak{f},b) + \omega s_t}{b\Delta_\omega} \right\} \tag{C.31}$$

The matrix $\mathbf{A}_z$ is the Markov transition matrix for z in the product space $\omega \times z$. Note that it is not indexed by z because the operator $\mathbf{A}_z$ is linear. The rows of both $\mathbf{A}_\omega\left[\mathfrak{f}_t, b_t\right]$ and $\mathbf{A}_z$ sum to zero to ensure that $\mathfrak{f}_t$ preserves mass.

The linearized system can be exactly expressed as (31) if the effect of $\mathfrak{f}$ on the interest rate is small. A sufficient condition is that the real interest rate is invariant to changes in the endowment share distribution, which would occur if consumption functions were linear in wealth. However, because the interest rate functional uses a consumption-based aggregator, in practice it is only necessary for the consumption function to be linear amongst high-wealth households, who consume relatively more of the aggregate endowment.

C.5 Uniqueness with Zero Surpluses

The government accumulation equation with zero surpluses is

$$\mathrm{d}b_t = [\mathbf{r}(\Omega_t)b_t]\mathrm{d}t \tag{C.32}$$

This implies a steady-state interest rate of $r^* = 0$ whenever $\mathbf{a}(0) > 0$, with an associated steady-state level of real debt given by $b^* \equiv \mathbf{a}(0)$. The first-order dynamics of this system around the steady-state are given by:

$$\mathrm{d}b_t = [b^* \partial_b \mathbf{r}(\Omega^*)]\mathrm{d}t \tag{C.33}$$

The last term is strictly positive due to household behaviour. Hence, the steady-state is locally saddle-path stable. Since $B_0 > 0$ is given, there exists a unique, finite value of P_0 such that the equilibrium converges back to the steady-state. There are also a continuum of stationary real equilibria with $P = \infty$, in which $r < \underline{r}$ and aggregate real debt is zero. This proves local uniqueness of the equilibrium. Conditions for global uniqueness are outlined in Online Appendix C.3.

C.6 Steady-State Welfare Comparison

We show that steady-states with higher real interest rates are Pareto ranked for any initial condition of assets a_{jt} and income z_{jt}. In particular, consider a particular profile of income shocks $\{z_{jt}\}_{t\geq 0}$ that induces a (realized) consumption and savings streams $\{c_{jt}, a_{jt}\}$ under a constant real interest rate r_L^*. This consumption plan can also be implemented at a higher interest rate $r_H^* > r_L^*$ for the same sequence of income shocks, since the change in savings in any given period will be:

$$da_{jt} = [(r_H^* - r_L^*)a_{jt}]dt \tag{C.34}$$

which is weakly positive for any given $a_{jt} > 0$ (recall that the surplus s^*, and hence taxes and transfers, are fixed and independent of the level of the real interest rate). Higher interest rates weakly expand the budget set of all households for any given a_{j0} and z_{j0}. This proves that a steady-state with r_H^* Pareto dominates r_L^*.[40]

[40] This proof strategy follows Aguiar et al. (2021), who construct robust Pareto-improving policies in the presence of capital accumulation.

C.7 Unique Steady State with Real Debt Reaction Rule

Our argument for uniqueness proceeds in three steps. First, we derive conditions for a unique steady-state. Second, we derive conditions for the steady-state to be saddle-path stable. This ensures local uniqueness. Finally, we consider whether explosive paths in debt can be ruled out globally. This ensures global uniqueness.

Steady-State Uniqueness. Suppose the government follows a fiscal rule of the form:

$$s_t = s^* + \phi_b(b_t - b^*) \tag{C.35}$$

where $s^* = r^* b^*$ is consistent with any given point on the household demand curve, so that the tuple $(b^*, r^*) = (\mathbf{a}(r^*), r^*)$ with $r^* < 0$. The government accumulation equation is:

$$\mathrm{d}b_t = [r_t b_t - s_t]\mathrm{d}t \tag{C.36}$$

The null-clines of the government accumulation equation are then defined by the following function:

$$r(b) = \frac{s^* - \phi_b b^*}{b} + \phi_b \tag{C.37}$$

A sufficient condition for steady-state uniqueness is that this function is downwards sloping. This will ensure that it intersects the upwards sloping steady-state demand curve $\mathbf{a}(r)$ exactly once. The slope of this function is

$$\frac{\mathrm{d}r}{\mathrm{d}b} = -\frac{s^* - \phi_b b^*}{b^2} \tag{C.38}$$

$$= -\frac{r^* b^* - \phi_b b^*}{b^2} \tag{C.39}$$

which is strictly negative whenever $r^* > \phi_b$. Hence, $\phi_b < r^* < 0$ is sufficient for steady-state uniqueness.

Local Uniqueness. We now examine conditions for this fiscal rule to give rise to local uniqueness. Under our maintained assumptions on the dynamical system that obtain (31), local uniqueness amounts to checking whether the eigenvalues of the government accumulation equation are strictly positive. The equilibrium dynamics are:

$$db_t = [(\mathbf{r}(\Omega_t) - \phi_b)b_t - (r^* - \phi_b)b^*]dt \tag{C.40}$$

The first-order dynamics of this system around the steady-state are given by:

$$db_t = [\mathbf{r}(\Omega^*) - \phi_b + b^* \partial_b \mathbf{r}(\Omega^*)]dt \tag{C.41}$$

The last term is positive because of household behavior. The sum of the first two terms are positive under the condition $r^* > \phi_b$. This proves local uniqueness.

Global Uniqueness. We now show that explosive dynamics are incompatible with equilibrium. Online Appendix C.3 shows that a sufficient condition for explosive dynamics to be inconsistent with equilibrium is for real debt to grow at a rate greater than r_t. But this follows from Equation (C.40) and $\phi_b < 0$.

C.8 Unique Steady State with Real Rate Reaction Rule

Our argument for uniqueness proceeds in three steps, as before.

Steady-State Uniqueness. Suppose the government follows a fiscal rule of the form:

$$s_t = s^* + \phi_r (r_t - r^*) \tag{C.42}$$

where $s^* = r^* b^*$ is consistent with any given point on the household demand curve, so that the tuple $(b^*, r^*) = (\mathbf{a}(r^*), r^*)$ with $r^* < 0$. The government accumulation equation is:

$$\mathrm{d}b_t = [r_t b_t - s_t]\mathrm{d}t \tag{C.43}$$

The null-clines of the government accumulation equation are then defined by the following function:

$$r(b) = \frac{(b^* - \phi_r) r^*}{b - \phi_r} \tag{C.44}$$

Our goal is to obtain an upward sloping function for the null-cline that intersects the r-axis above $\mathbf{a}(r)$. This will ensure that it intersects the upwards sloping steady-state demand curve $\mathbf{a}(r)$ exactly once, as in Figure 5b. The slope of this function is

$$\frac{\mathrm{d}r}{\mathrm{d}b} = -\frac{(b^* - \phi_r) r^*}{(b - \phi_r)^2} \tag{C.45}$$

which is strictly positive whenever $b^* > \phi_r$. We also want the null-cline to intersect the r-axis at a negative real interest rate that is greater than $\underline{r}$ (c.f. Figure 5b). This occurs if

$$\phi_r < \frac{s^*}{r^* - \mathbf{a}^{-1}(0)} \tag{C.46}$$

Local Uniqueness. We now examine conditions for this fiscal rule to yield local uniqueness. Under our maintained assumptions on the dynamical system that obtain (31), local uniqueness amounts to checking whether the eigenvalues of the government accumulation equation are strictly positive. The equilibrium dynamics are:

$$\mathrm{d}b_t = \mathbf{r}(\Omega_t)(b_t - \phi_r) - (r^* - \phi_r)b^*]\mathrm{d}t \tag{C.47}$$

The first-order dynamics of this system around the steady-state are given by:

$$\mathrm{d}b_t = [\mathbf{r}(\Omega^*) + (b^* - \phi_r)\partial_b \mathbf{r}(\Omega^*)]dt \tag{C.48}$$

Note that at the top-right steady-state, we must have

$$\mathbf{r}'(\Omega^*) > -\frac{r^*}{b^*} \tag{C.49}$$

which ensures that a sufficient condition for the right-hand side of (C.48) to be positive is $\phi_r < 0$. Hence, $\phi_r < 0$ is a sufficient condition for local uniqueness.

Global Uniqueness. We now show that explosive dynamics are incompatible with equilibrium. Online Appendix C.3 shows that a sufficient condition for explosive dynamics to be inconsistent with equilibrium is for real debt to grow at a rate greater than r_t. But this follows from Equation (C.40) and $\phi_r < 0$.

C.9 Local Dynamics with Interest Payment Reaction Rule

Steady-State Invariance. Suppose the government follows the fiscal rule:

$$s_t = s^* + \phi_s(r_t b_t - r^* b^*) \tag{C.50}$$

where $s^* = r^* b^*$ is consistent with any given point on the household demand curve, so that the tuple $(b^*, r^*) = (\mathbf{a}(r^*), r^*)$ with $r^* < 0$. The government accumulation equation is:

$$db_t = [r_t b_t - s_t]dt \tag{C.51}$$

The null-clines of the government accumulation equation are then defined by the following function:

$$r(b) = \frac{s^* - \phi_s r^* b^*}{b - \phi_s b^*} = \frac{s^*}{b} \tag{C.52}$$

which shows that the steady-states are unchanged. Hence, there is no scope for this fiscal rule to eliminate steady-state multiplicity.

Local Dynamics. The dynamics of government debt are given by

$$db_t = (1 - \phi_s)\left(\mathbf{r}(\Omega_t) b_t - s^*\right) dt \tag{C.53}$$

It follows that the stability properties of the two-steady states in the baseline case with $\phi_s = 0$ are reversed when $\phi_s > 1$.

D Model With Foreign Demand for Debt

We assume that there exists a foreign sector that is populated by a representative household. The foreign representative household derives utility over real consumption streams and real debt holdings in terms of US goods.[41] Preferences over foreign consumption and bonds are given by

$$u\left(c_t, d_t\right) = \frac{c_t^{1-\gamma}}{1-\gamma} + \tilde{\zeta} \frac{d_t^{1-\theta}}{1-\theta}$$

with $\gamma \geq 0$ and $\theta \geq 0$. The parameter $\tilde{\zeta} > 0$ parameterizes the payoff derived from real bond holdings. Households' rate of time preference is $\tilde{\rho}$. We assume the foreign sector grows at the same rate g as the domestic economy, thereby allowing for the existence of a balanced growth path. The household's growth-adjusted discount rate is therefore $\rho := \tilde{\rho} - (1-\gamma)\, g$. In addition, we define $r_t := i_t - \pi_t - g$.

The household's budget constraint in real and stationary terms is therefore

$$\mathrm{d}d_t = [r_t d_t + y^f - c_t]\mathrm{d}t \tag{D.54}$$

[41] Concretely, the foreign sector derives utility from nominal bonds in dollars divided by the US price level: $B_t/P_t^U S$. This is equivalent to holding real debt in terms of the foreign sector good $(P_t^F B_t/(P_t^F B_t))$ where P_t^F is the foreign price level. The implicit assumption here is that the final good is tradable, so that the exchange rate is constant.

where $y^f > 0$ is the foreign household's endowment of the consumption good. Foreign real consumption and real debt holdings must satisfy the following Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma}\left(r_t - \rho + \tilde{\zeta}\frac{d_t^{-\theta}}{c_t^{-\gamma}}\right)$$

In steady-state, the aggregate consumption of the foreign sector is y^f. Hence, for a given interest rate r^*, we must have

$$0 = r^* - \rho + \frac{\tilde{\zeta}}{y^f} d^{-\theta}$$

It follows that foreign sector demand for US government debt is given by

$$\mathbf{d}(r^*) \quad = \quad \left(\frac{\rho - r^*}{\zeta}\right)^{-\frac{1}{\theta}}$$

where we let $\zeta := \tilde{\zeta}/y_f$. Note that the relationship between foreign debt holdings and real interest rates can be written as:

$$\log d = \zeta + \frac{1}{\theta}\log(r^* - \rho)$$

so a large θ means more inelastic demand.

Consider the limits of this function: as $r^* \to -\infty$, $d \to 0$ and as $r^* \to \rho$, $d \to \infty$. We want to argue that when the foreign demand is inelastic enough (θ large), we can get rid of the high inflation steady-state. From Figure 5c, it is clear that what we need is the bond supply function $\mathbf{b}(r)$ to lie above the total bond demand function $\mathbf{a}(r) + \mathbf{d}(r)$ as $r \to -\infty$. This condition can be re-written as:

$$\lim_{r \to \infty} \quad \frac{\mathbf{b}(r)}{\mathbf{a}(r) + \mathbf{d}(r)} < 1$$

Or, equivalently

$$\lim_{b \to 0} \quad \frac{r^g(b)}{r^h(b) + r^f(b)} < 1$$

where $r^g(b)$, $r^h(b)$, and $r^f(b)$ denotes the inverse debt demand functions for the government, domestic households, and the foreign sector, respectively. Substituting for these functional forms yields

$$\lim_{b\to 0}\frac{\frac{s}{b}}{\mathbf{a}^{-1}(b)+(\rho-\zeta b^{-\theta})}$$

Online Appendix C.1 shows that there exists a finite real interest rate $\underline{r}$ such that $\mathbf{a}(\underline{r})=0$. We may apply L'Hopital's rule to obtain

$$\lim_{b\to 0}\frac{-s}{b^2\left(\kappa+\theta b^{-\theta-1}\right)}<1$$

where κ is a finite positive constant. This inequality is satisfied if and only if

$$-s<\theta b^{1-\theta}$$

Recall that $s<0$ so the LHS is a positive finite number. As long as $\theta>1$ the right-hand-side converges to infinity as $b\to 0$ which satisfies the inequality. Hence, the condition we need in order to obtain steady-state uniqueness is $\theta>1$ which implies that the foreign demand has to be inelastic enough.

E Extended Model for Quantitative Analysis

E.1 Model With Borrowing

In this section, we describe how the model is consistent with a non-zero lower bound on real household assets and costly borrowing. Households face a borrowing limit expressed in real terms:

$$\frac{A_{jt}}{P_t}\geq\underline{\tilde{a}}_t \tag{E.55}$$

In order for the borrowing constraint to be consistent with balanced growth, we assume that $\underline{\tilde{a}}_t$ grows at the rate of real output, $\underline{\tilde{a}}_t=y_0e^{gt}\underline{a}$ for some $\underline{a}<0$. Note that this implies that $a_{jt}\geq\underline{a}$. Furthermore, we assume that borrowing is costly. Households face a wedge $\vartheta\geq 0$ on the real interest rate when borrowing, so that the interest rate they pay on debt is $\vartheta+r_t$. This borrowing wedge creates deadweight loss in output that is equal to the wedge multiplied by the amount of assets borrowed. This implies that aggregate consumption is slightly less than output (a difference around 0.3% of steady-state output in our calibration). The associated boundary condition for the HJB equation (9) is:

$$\partial_a V_t(0,z) \geq (z - \tau_t(z) - (r_t + \vartheta)\underline{a})^{-\gamma} \tag{E.56}$$

The government accumulation equation continues to follow equation (18), with the understanding that $b_t \geq 0$, so that the government can lend, but not borrow.

E.2 Model With Long-Term Debt

The government now issues two securities: short-term debt B_t^s that pays a nominal rate i_t, and long-term debt B_t^l. Long-term debt takes the form of depreciating consoles that depreciate at a rate $\delta > 0$, and that yield a flow coupon payment of $\chi > 0$ as in Cochrane (2001). We let q_t denote the market value of this long-term bond. The government's budget constraint can be written as:

$$dB_t^s + q_t dB_t^l = [iB_t^s + (\chi - \delta q_t)B_t^l - P_t s_t]dt \tag{E.57}$$

The intuition for this equation is as follows. The right-hand side is the government's nominal deficit that consists of the primary deficit $-P_t s_t$, interest payments on short-term debt iB_t^s, and coupon payments plus redemption of long-term debt $(\chi - \delta q_t)B_t^l$. Whenever the deficit is greater than zero, the government must issue additional debt. It can do so either by issuing additional short-term debt or by issuing additional long-term debt at the price of q_t.

Similarly, we may define the nominal short- and long-term debt holdings of household j at time t as A_{jt}^s and A_{jt}^l, respectively. The household budget constraint becomes

$$dA_{jt}^s + q_t dA_{jt}^l = [i_t A_{jt}^s + (\chi - \delta q_t)A_{jt}^l + (z_{jt} - \tau_t(z_{jt}))P_t y_t - P_t \tilde{c}_{jt}]dt \tag{E.58}$$

We define the market value of total government debt outstanding as $B_t := B_t^s + q_t B_t^l$ and the total value of household assets as $A_{jt} := A_{jt}^s + q_t A_{jt}^l$. We also define de-trended real debt and assets as in the main text:

$$b_{jt} = \frac{B_t}{P_t} \quad \text{and} \quad a_{jt} = \frac{A_{jt}}{P_t} \tag{E.59}$$

The next proposition demonstrates that there is maturity structure irrelevance for government debt: b_t and a_{jt} are the only state variables in this economy.

Proposition 4. *The household budget constraint follows (6) and the real government budget constraint follows (18) for $t > 0$. Moreover, the price of long-term debt satisfies the following differential equation for $t > 0$:*

$$\frac{\dot{q}_t}{q_t} + \frac{\chi - \delta q_t}{q_t} = i_t \tag{E.60}$$

Proof. See Not For Publication Appendix I □

This proof shows that an economy with long-term debt collapses into an economy with short-term debt in the absence of uncertainty. Equation (E.60) is an arbitrage relationship between short- and long-term debt. In equilibrium, households are indifferent between the two assets. Hence, long-term debt will only matter for inflation dynamics insofar there is an unanticipated change in nominal rates i_t. Equation (E.60) is a forward-looking equation.

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